QUADRATURE DOMAINS AND KERNEL FUNCTION ZIPPING

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Abstract. It is proved that quadrature domains are ubiquitous in a very strong sense in the realm of smoothly bounded multiply connected domains in the plane. In fact, they are so dense that one might as well assume that any given smooth domain one is dealing with is a quadrature domain, and this allows access to a host of strong conditions on the classical kernel functions associated to the domain. Following this string of ideas leads to the discovery that the Bergman kernel can be "zipped" down to a strikingly small data set.

It is also proved that the kernel functions associated to a quadrature domain must be algebraic.

1. Introduction. In this paper, we will refine results of B. Gustafsson in light of recent results in [8] about the complexity of the classical kernels functions to show that quadrature domains in the plane are so dense that one cannot possibly devise a test to determine if a given smooth domain is a quadrature domain. The combined methods of Gustafsson [12] and [8] will also yield a method to "zip" the Bergman kernel down to a very small data set consisting of finitely many complex numbers plus the boundary values of a *single* holomorphic function, which I would venture to christen a *Gustafsson function.* These results are all a natural outgrowth of the work of Aharonov and Shapiro [1] and Shapiro [15], and one consequence of the Aharonov-Shapiro theorem that Ahlfors maps associated to quadrature domains are algebraic will be that the Bergman and Szegő kernels associated to a quadrature domain are algebraic functions.

For the purposes of this paper, we shall call an n-connected domain Ω in the plane such that no boundary component is a point a *quadrature domain* if there exist finitely many points $\{w_j\}_{j=1}^N$ in the domain and non-negative integers n_j such that complex numbers c_{jk} exist satisfying

$$\int_\Omega f\ dA = \sum_{j=1}^N \sum_{k=0}^{n_j} c_{jk} f^{(k)}(w_j) \tag{1.1}$$

for every function f in the Bergman space of square integrable holomorphic functions on Ω. Here, dA denotes Lebesgue area measure. Many of our results require the function $h(z) \equiv 1$ to be in the Bergman space, and so we shall often also assume that the domain under study has *finite area.* We remark that there are results of Sakai that show that, under certain weaker assumptions, a quadrature domain

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does have finite area, so some of our results could be stated with weaker hypotheses. (See [12] for an explanation of how Sakai's results relate to the type of quadrature domains we study here.)

The Ahlfors map associated to a point a in an n-connected domain Ω such that no boundary component is a point is the holomorphic function f_a such that f_a maps Ω into the unit disc maximizing the quantity $|f_a'(a)|$ with $f_a'(a)$ real and positive. This map is an n-to-one (counting multiplicities) proper holomorphic mapping of Ω onto the unit disc.

Quadrature domains have particularly simple kernel functions, as our first theorem shows.

Theorem 1.1. *Suppose that Ω is an n-connected quadrature domain of finite area in the plane such that no boundary component is a point. Then the Bergman kernel function $K(z,w)$ associated to Ω is a rational combination of two Ahlfors maps f_a and f_b in the sense that $K(z,w)$ is a rational combination of $f_a(z)$, $f_b(z)$, $\overline{f_a(w)}$, and $\overline{f_b(w)}$. The same can be said of the square $S(z,w)^2$ of the Szegő kernel. Furthermore, the classical functions F_j' are rational functions of f_a and f_b.*

The functions F_j' are defined precisely in §2.

Aharonov and Shapiro [1] proved that Ahlfors maps associated to quadrature domains are algebraic. Hence, Theorem 1.1 yields that quadrature domains also have algebraic kernel functions.

Theorem 1.2. *Suppose that Ω is an n-connected quadrature domain in the plane of finite area such that no boundary component is a point. The Bergman and Szegő kernel functions associated to Ω are algebraic functions. The classical functions F_j' are also algebraic.*

Similar statements to Theorems 1.1 and 1.2 hold for the Poisson kernel and first derivative of the Green's function. These results follow from formulas appearing in [6] and we do not spell them out here.

Ahlfors maps extend to the double (as described in §2 of this paper), and it follows that, under the hypotheses of Theorem 1.1, the Bergman kernel function extends meromorphically to the double $\widehat{\Omega}$ of Ω, and is therefore a rational combination of any two functions that generate the meromorphic functions on the double, i.e., any two functions that form a primitive pair for the double. The Bergman kernel always extends to the double as a meromorphic differential, but extending as a meromorphic functions is rather unusual behavior for the kernel. This condition leads to a number of other strong conclusions that we now begin to enumerate.

Theorem 1.3. *Suppose that Ω is an n-connected quadrature domain of finite area in the plane such that no boundary component is a point. If f is any proper holomorphic mapping of Ω onto the unit disc, then f' extends to the double of Ω as a meromorphic function.*

Under the assumptions of Theorem 1.3, since both f and f' extend to the double, they are algebraically dependent, i.e., there exists an irreducible polynomial $P(z,w)$ on $\mathbb{C}^2$ such that $P(f',f)\equiv 0$ on Ω. This was proved by other means by Aharonov and Shapiro in [1]. It is proved in [8] that the condition $P(f',f)=0$ has a number of implications, two of which are that the kernel functions are generated by only two functions and that the kernel functions extend to a compact Riemann surface. In the setting of Theorem 1.3, however, we have the stronger conclusion that the

Bergman kernel extends to the compact Riemann surface which is the double of Ω, and that it is generated by any two functions that form a primitive pair for the double.

When combined with the main theorem of [9], Theorem 1.3 yields that the infinitesimal Carathéodory metric associated to an n-connected quadrature domain of finite area such that no boundary component is a point is a real algebraic function which is a rational combination of two Ahlfors maps and their conjugates.

It is shown in [7] that, under the assumptions of Theorem 1.3, if f is any proper holomorphic map of Ω onto the unit disc, then it is possible to find an Ahlfors map f_b such that f and f_b extend to the double and generate the meromorphic functions on the double. Hence, it follows that $f' = R(f, f_b)$ for some rational function. Also, we may conclude that, given a proper map f, the Bergman kernel is a rational combination of f and some other Ahlfors map. Furthermore, since both f' and f_b' extend meromorphically to the double, we deduce the next rather odd sounding theorem.

Theorem 1.4. *Suppose that Ω is an n-connected quadrature domain of finite area in the plane such that no boundary component is a point. If H is any meromorphic function on Ω that extends meromorphically to the double of Ω, then H' also extends meromorphically to the double of Ω. Furthermore, H is algebraic.*

Of course if H is meromorphic on Ω, then H' is meromorphic on Ω. The content of the theorem is that H' *extends* meromorphically to the double if H does.

The property in Theorem 1.4 turns out to characterize a class of generalized quadrature domains, and we explore this line of reasoning in §5.

When we combine the ideas used in the proofs of the results above with those of Aharonov and Shapiro [1] and Gustafsson [12], we can show that the kernel functions associated to quadrature domains are particularly simple when restricted to the boundary.

Theorem 1.5. *Suppose that Ω is an n-connected quadrature domain in the plane of finite area such that no boundary component is a point. The Bergman kernel $K(z, w)$ and the square $S(z, w)^2$ of the Szegő kernel are rational functions of z, $\bar{z}$, w, and $\bar{w}$ on $b\Omega \times b\Omega$ minus the boundary diagonal. The functions $F_j(z)$ are rational functions of z and $\bar{z}$ when restricted to the boundary. Furthermore, the unit tangent vector function $T(z)$ is such that $T(z)^2$ is a rational function of z and $\bar{z}$ for $z \in b\Omega$.*

I had conjectured in [5] that every n-connected domain in the plane such that no boundary component is a point is conformally equivalent to a domain with algebraic kernel functions. Jeong and Taniguchi [14] recently verified this conjecture. Since Gustafsson proved in [12] that every such domain is conformally equivalent to a quadrature domain of finite area, Theorem 1.2 gives an alternate way of seeing that every n-connected domain in the plane such that no boundary component is a point is conformally equivalent to a domain with algebraic kernel functions.

We remark here that, although the last part of Theorem 1.5 might seem to suggest that the Bergman kernel associated to a quadrature domain could be a simple rational function of some kind, it can never happen that the Bergman kernel is a rational function in the setting of multiply connected domains (see [4]).

The main results of this paper together with Gustafsson's theorem that any finitely connected domain in the plane such that no boundary component is a point

is conformally equivalent to a smoothly bounded quadrature domain suggests that quadrature domains might serve to play a role in the multiply connected setting similar to that played by the unit disc for simply connected regions.

Quadrature domains with smooth boundaries are particularly appealing and we can refine arguments of Gustafsson to prove the next theorem, which shows that very fine modifications can be made to any smoothly bounded domain to make it a quadrature domain.

Theorem 1.6. *Suppose that Ω is a bounded n-connected domain whose boundary consists of n non-intersecting C^∞ smooth simple closed curves. There is a meromorphic function g on the double of Ω which has no poles on $\overline{\Omega}$ such that g is as close to the identity map in $C^\infty(\overline{\Omega})$ as desired. The domain given by $g(\Omega)$ is a quadrature domain which is C^∞ close to Ω and conformally equivalent to Ω.*

The analytic objects attached to the quadrature domain $g(\Omega)$ have the strong extension properties given in the preceding theorems and they are C^∞ close to the analytic objects attached to Ω. In particular, Aharonov and Shapiro [1] (with some refinements by Gustafsson [12]) showed that the boundary of $g(\Omega)$ is an algebraic curve minus perhaps finitely many points. Thus, the proof of Theorem 1.6 will yield a concrete method to approximate in C^∞ a non-intersecting group of n simple closed C^∞ curves by an algebraic curve. In fact, the algebraic curve can be described by $|f(z)|^2 = 1$ where f is any Ahlfors map.

We describe in §4 how the Bergman kernel can be recovered from the boundary values of g in a very simple and efficient manner.

Gustafsson proved that the function g in Theorem 1.6 maps Ω to a quadrature domain. We shall show that $g(z)$ can be taken to be a linear combination of functions of the form $S(z,b)/L(z,a)$ where $S(z,b)$ is the Szegő kernel and $L(z,a)$ is the Garabedian kernel, and b ranges over a small open subset of Ω while a is fixed. Consequently, we shall be able to restrict the points w_j in the defining property (1.1) of the quadrature domain $g(\Omega)$ to a small set. We shall also be able to specify the numbers n_j in rather surprising ways. In particular, we shall be able to stipulate that $n_j = 1$ for each j. Thus, any smooth domain is conformally equivalent to a nearby quadrature domain where the simple point masses are contained in an arbitrarily small arbitrary disc that is compactly contained in Ω. Another way to state this is to say that it is possible to strongly approximate the two dimensional field generated by a uniform charge density on a smoothly bounded plate with holes by point charges at finitely many points in an arbitrarily small open subset of the plate. This result is stated precisely in the following theorem.

Theorem 1.7. *Suppose that Ω is a bounded n-connected domain whose boundary consists of n non-intersecting C^∞ smooth simple closed curves. Let $D_\epsilon(w_0)$ be any disc which is compactly contained in Ω. There is a quadrature domain which is C^∞ close to Ω and conformally equivalent to Ω such that the point masses appearing in (1.1) all fall in $D_\epsilon(w_0)$ and have weight $n_j = 1$. Furthermore, given w_0 in Ω, there is a quadrature domain which is C^∞ close to Ω and conformally equivalent to Ω such that w_0 is the only point mass appearing in (1.1), i.e., $N = 1$ in (1.1).*

In §6 of this paper, we show how many of the same ideas can be extended to quadrature domains with respect to boundary arc length measure.

2. Preliminaries. It is a standard construction in the theory of conformal mapping to show that an n-connected domain Ω in the plane such that no boundary

component is a point is conformally equivalent via a map Φ to a bounded domain $\widetilde{\Omega}$ whose boundary consists of n simple closed C^∞ smooth real analytic curves. Since such a domain $\widetilde{\Omega}$ is a bordered Riemann surface, the double of $\widetilde{\Omega}$ is an easily realized compact Riemann surface. We shall say that an analytic or meromorphic function h on Ω *extends meromorphically to the double of* Ω if $h \circ \Phi^{-1}$ extends meromorphically to the double of $\widetilde{\Omega}$. Notice that whenever Ω is itself a bordered Riemann surface, this notion is the same as the notion that h extends meromorphically to the double of Ω. We shall say that two functions G_1 and G_2 extend to the double and generate the meromorphic functions on the double of Ω, and that they therefore form a primitive pair for the double of Ω, if $G_1 \circ \Phi^{-1}$ and $G_2 \circ \Phi^{-1}$ extend to the double of $\widetilde{\Omega}$ and form a primitive pair for the double of $\widetilde{\Omega}$ (see Farkas and Kra [11] for the definition and basic facts about primitive pairs).

It is proved in [7] that if Ω is an n-connected domain in the plane such that no boundary component is a point, then almost any two distinct Ahlfors maps f_a and f_b generate the meromorphic functions on the double of Ω. It is also proved that any proper holomorphic mapping from Ω to the unit disc extends to the double of Ω.

Suppose that Ω is a bounded n-connected domain whose boundary consists of n non-intersecting C^∞ smooth simple closed curves. The Bergman kernel $K(z, w)$ associated to Ω is related to the Szegő kernel via the identity

$$K(z,w) = 4\pi S(z,w)^2 + \sum_{i,j=1}^{n-1} A_{ij} F_i'(z) \overline{F_j'(w)}, \tag{2.1}$$

where the functions $F_i'(z)$ are well known classical functions of potential theory described as follows. The harmonic function ω_j which solves the Dirichlet problem on Ω with boundary data equal to one on the boundary curve γ_j and zero on γ_k if $k \neq j$ has a multivalued harmonic conjugate. Let γ_n denote the outer boundary curve. The function $F_j'(z)$ is a single valued holomorphic function on Ω which is locally defined as the derivative of $\omega_j + iv$ where v is a local harmonic conjugate for ω_j. The Cauchy-Riemann equations reveal that $F_j'(z) = 2(\partial\omega_j/\partial z)$.

The Bergman and Szegő kernels are holomorphic in the first variable and anti-holomorphic in the second on $\Omega \times \Omega$ and they are hermitian, i.e., $K(w,z) = \overline{K(z,w)}$. Furthermore, the Bergman and Szegő kernels are in $C^\infty((\overline{\Omega} \times \overline{\Omega}) - \{(z,z) : z \in b\Omega\})$ as functions of (z,w) (see [2, page 100]).

We shall also need to use the Garabedian kernel $L(z,w)$, which is related to the Szegő kernel via the identity

$$\frac{1}{i} L(z,a) T(z) = S(a,z) \qquad \text{for } z \in b\Omega \text{ and } a \in \Omega \tag{2.2}$$

where $T(z)$ represents the complex unit tangent vector at z pointing in the direction of the standard orientation of $b\Omega$. For fixed $a \in \Omega$, the kernel $L(z,a)$ is a holomorphic function of z on $\Omega - \{a\}$ with a simple pole at a with residue $1/(2\pi)$. Furthermore, as a function of z, $L(z,a)$ extends to the boundary and is in the space $C^\infty(\overline{\Omega} - \{a\})$. In fact, $L(z,w)$ is in $C^\infty((\overline{\Omega} \times \overline{\Omega}) - \{(z,z) : z \in \overline{\Omega}\})$ as a function of (z,w) (see [2, page 102]). Also, $L(z,a)$ is non-zero for all (z,a) in $\overline{\Omega} \times \Omega$ with $z \neq a$ and $L(a,z) = -L(z,a)$ (see [2, page 49]).

For each point $a \in \Omega$, the function of z given by $S(z,a)$ has exactly $(n-1)$ zeroes in Ω (counting multiplicities) and does not vanish at any points z in the boundary of Ω (see [2, page 49]).

Given a point $a \in \Omega$, the Ahlfors map f_a associated to the pair (Ω, a) is a proper holomorphic mapping of Ω onto the unit disc. It is an n-to-one mapping (counting multiplicities), it extends to be in $C^\infty(\overline{\Omega})$, and it maps each boundary curve γ_j one-to-one onto the unit circle. Furthermore, $f_a(a) = 0$, and f_a is the unique function mapping Ω into the unit disc maximizing the quantity $|f_a'(a)|$ with $f_a'(a) > 0$. The Ahlfors map is related to the Szegő kernel and Garabedian kernel via (see [2, page 49])

$$f_a(z) = \frac{S(z,a)}{L(z,a)}. \tag{2.3}$$

Note that $f_a'(a) = 2\pi S(a,a) \neq 0$. Because f_a is n-to-one, f_a has n zeroes. The simple pole of $L(z,a)$ at a accounts for the simple zero of f_a at a. The other $n-1$ zeroes of f_a are given by the $(n-1)$ zeroes of $S(z,a)$ in $\Omega - \{a\}$.

When Ω does not have smooth boundary, we define the kernels and domain functions above as in [6] via a conformal mapping to a domain with real analytic boundary curves.

3. Proofs of the theorems. If Ω is an n-connected quadrature domain of finite area in the plane such that no boundary component is a point, then the Bergman kernel function associated to Ω satisfies an identity of the form

$$1 \equiv \sum_{j=1}^{N} \sum_{m=0}^{n_j} c_{jm} K^{(m)}(z, w_j) \tag{3.1}$$

where $K^{(m)}(z,w)$ denotes $(\partial^m/\partial \bar{w}^m)K(z,w)$ and the points w_j are the points that appear in the characterizing formula (1.1) of quadrature domains. This observation is usually attributed to Avci in his unpublished Stanford PhD thesis. It can be seen by noting that the inner product of an analytic function against the function $h(z) \equiv 1$ and against the sum on the right hand side of (3.1) agree for all functions in the Bergman space. Hence the two functions must be equal. Note that we must assume that Ω has finite area here just so that $h(z) \equiv 1$ is in the Bergman space.

Proof of Theorem 1.1. Since the Bergman kernel is equal to $(\partial^2/\partial z \partial \bar{w})$ of the Green's function, functions that are of the form of the right hand side of (3.1) belong to the the class $\mathcal{A}$ of [8, p. 20]. Hence, the function $A(z) \equiv 1$ belongs to $\mathcal{A}$. Theorem 2.3 of [8] states that if G_1 and G_2 are any two meromorphic functions on Ω that extend to the double of Ω to form a primitive pair and if $A(z)$ is any function from the class $\mathcal{A}$ other than the zero function, then the Bergman kernel associated to Ω can be expressed as

$$K(z,w) = A(z)\overline{A(w)} R_1(G_1(z), G_2(z), \overline{G_1(w)}, \overline{G_2(w)})$$

where R_1 is a complex rational function of four complex variables. Similarly, the Szegő kernel can be expressed as

$$S(z,w)^2 = A(z)\overline{A(w)} R_2(G_1(z), G_2(z), \overline{G_1(w)}, \overline{G_2(w)})$$

where R_2 is rational, and the functions F_j' can be expressed

$$F_j'(z) = A(z)R_3(G_1(z), G_2(z))$$

where R_3 is rational. Furthermore, every proper holomorphic mapping of Ω onto the unit disc is a rational combination of G_1 and G_2. It therefore follows now that the Bergman kernel is a rational combination of any two meromorphic functions on Ω that extend to the double to form a primitive pair. Since almost any two distinct Ahlfors maps form a primitive pair (see [7]), the proof of Theorem 1.1 is complete.

Proof of Theorem 1.3. Suppose that Ω is an n-connected quadrature domain in the plane such that no boundary component is a point and suppose that f is a proper holomorphic mapping of Ω onto the unit disc. We may compose f with a Möbius transformation φ so that $F = \varphi \circ f$ has only simple zeroes at, say $a_1, a_2, \ldots, a_N$, where N is the order of the proper map f. It is proved in [2, p. 65] that the Bergman kernel transforms under this proper map according to

$$F'(z)K_D(F(z), 0) = \sum_{n=1}^{N} K(z, a_n)/\overline{F'(a_n)}$$

where $K_D(z, w) = \pi^{-1}(1 - z\bar{w})^{-2}$ is the Bergman kernel for the unit disc. Notice that $K_D(z, 0) \equiv \pi^{-1}$, and so it follows that $F'(z)$ is given by a linear combination of functions of the form $K(z, a_n)$, and thus F' extends to the double of Ω by Theorem 1.1. But $F'(z) = f'(z)\varphi'(f(z))$, and since φ is rational and f extends to the double of Ω, it now follows that $f'(z)$ extends to the double of Ω. The proof is complete.

Proof of Theorem 1.5. In the setting of Theorem 1.5, Gustafsson [12] generalized a result of Aharonov and Shapiro [1] to prove that the boundary of Ω is given by an algebraic curve and that there exists a function $H(z)$ which is meromorphic on Ω with continuous boundary values such that $H(z) = \bar{z}$ on $b\Omega$. Let $G(z) = z$. Gustafsson proved that $H(z)$ and $G(z)$ extend to the double of Ω to form a primitive pair. Hence, there exists an irreducible polynomial $P(z, w)$ on $\mathbb{C}^2$ such that $P(H(z), G(z)) \equiv 0$ on Ω. This shows that $H(z)$ is an algebraic function of z. We know that the Bergman kernel is generated by z and $H(z)$. Hence, this gives another way to see that the Bergman kernel is algebraic. It is proved in [5] that if the Bergman kernel is algebraic, then so is the Szegő kernel, all proper holomorphic maps onto the unit disc, and the classical functions F_j'.

Now since the kernels $K(z, w)$ and $S(z, w)^2$ and the proper holomorphic maps to the unit disc and the functions F_j are all generated by $G(z)$ and $H(z)$, and since these functions are equal to z and $\bar{z}$, respectively on the boundary, we may deduce most of the rest of the claims made in Theorem 1.5. To finish the proof, note that identity (2.2) yields that

$$T(z)^2 = -\frac{S(a, z)^2}{L(z, a)^2}$$

where a is an arbitrary point chosen and fixed in Ω. The function $S(z, a)^2$ is a rational function of z and $\bar{z}$ on the boundary. Identity (2.3) yields that $L(z, a)^2 = S(z, a)^2/f_a(z)^2$, and so $L(z, a)^2$ is also a rational function of z and $\bar{z}$ on the boundary. Finally, it follows that $T(z)^2$ is a rational function of z and $\bar{z}$.

We remark that, since the antiholomorphic Schwarz reflection function $S(z)$ across a real analytic boundary curve of Ω satisfies $\overline{f(S(z))} = 1/f(z)$ when f is a proper holomorphic mapping onto the unit disc, it follows that $S(z)$ is algebraic whenever proper holomorphic maps to the disc are.

Proof of Theorem 1.6. Suppose that Ω is a bounded n-connected domain whose boundary consists of n non-intersecting C^∞ smooth simple closed curves. I proved in [10] (see also [2, p. 29]) that the complex linear span of the set of functions of z given by $\{S(z, b) : b \in \Omega\}$ is dense in $A^\infty(\Omega)$, the subset of $C^\infty(\overline{\Omega})$ consisting of holomorphic functions on Ω. The proof given there is constructive. It is proved in [3] that there is a dense open set of points a in Ω such that $S(z, a)$ has $n-1$ simple zeroes as a function of z. Fix such a point a and let $a_1, a_2, \ldots, a_{n-1}$ denote the zeroes of $S(z, a)$. The functions of z given by $S(z, b)/L(z, a)$ extend meromorphically to the double $\widehat{\Omega}$ of Ω because identity (2.2) shows that $S(z, b)/L(z, a)$ agrees with the conjugate of $L(z, b)/S(z, a)$ on the boundary of Ω. Let $R(z)$ denote the antiholomorphic reflection function which maps Ω to its reflected copy in the double. Notice that the extended function has no poles in $\overline{\Omega}$ and, if b is not equal to any of the zeroes a_j, then it has simple poles at $R(b)$ and the points $\{R(a_j)\}_{j=1}^{n-1}$.

The function $H(z)$ which is equal to $(z-a)L(z, a)$ for $z \in \Omega$, $z \neq a$, and equal to $1/2\pi$ at $z = a$ is in $A^\infty(\Omega)$. Hence, we may find finitely many points b_j in Ω such that a linear combination $\sum_{j=1}^{N} c_j S(z, b_j)$ is as close to $H(z)$ in $A^\infty(\Omega)$ as desired. Now the function $g(z)$ given by

$$a + \sum_{j=1}^{N} c_j S(z, b_j)/L(z, a)$$

extends to be a meromorphic function on the double of Ω which is close in $C^\infty(\overline{\Omega})$ to the identity function. It is this function g that we wish to call a *Gustafsson function*. We shall use it in the next section to zip the Bergman kernel.

Gustafsson proved in [12] that the poles of the function g on the reflected copy of Ω in the double of Ω reflect back to the points in Ω that map under g to points that appear in the quadrature identity for $g(\Omega)$. Hence, the points in $g(\Omega)$ that would appear in the quadrature identity (1.1) for $g(\Omega)$ are among the images under g of the points $a_1, \ldots, a_{n-1}$ and $b_1, \ldots, b_N$. We shall refine the proof above to get more control over these points momentarily.

Proof of Theorem 1.7. The proof just given of Theorem 1.6 can be altered so that the points b_j fall in a very small set and so that all the integers n_j in the quadrature identity for $g(\Omega)$ are equal to one. Indeed, let $D_\epsilon(w_0)$ be any disc which is compactly contained in Ω. Choose a in Ω such that $S(z, a)$ has $n-1$ simple zeroes as a function of z. Let $a_1, a_2, \ldots, a_{n-1}$ denote the zeroes of $S(z, a)$ and let $a_0 = a$. We shall now repeat the argument above, but we shall restrict the points b to be in $D_\epsilon(w_0)$. Indeed, we now claim that the complex linear span $\mathcal{L}$ of $\{S(z, b) : b \in D_\epsilon(w_0)\}$ is dense in $A^\infty(\Omega)$. The dual space $A^{-\infty}(\Omega)$ of $A^\infty(\Omega)$ is described in [2, p. 117] (see also [10]). If $\mathcal{L}$ were not dense in $A^\infty(\Omega)$, then there would be a function $h \in A^{-\infty}(\Omega)$ which is not the zero function, but which is orthogonal to $\mathcal{L}$ with respect to the non-degenerate pairing which extends the usual L^2 inner product on Ω. But the function $H(b)$ given as $\langle h, S(\cdot, b)\rangle$ is a holomorphic function of b on Ω. Thus, if h is orthogonal to $\mathcal{L}$, then $H(b)$ vanishes on $D_\epsilon(w_0)$, and is therefore

zero on all of Ω. This shows that h is orthogonal to $S(z,b)$ for all b in Ω, and we know that these functions span a dense subset of $A^\infty(\Omega)$. Hence, $h \equiv 0$, and this contradiction yields that $\mathcal{L}$ must be dense.

The function $H(z)$ given by

$$H(z) = zf_a(z)L(z,a)$$

for $z \in \Omega$, $z \neq a$, and equal to $af_a'(a)/2\pi$ at $z = a$ is in $A^\infty(\Omega)$. Hence, we may find finitely many points b_j in $D_\epsilon(w_0)$ such that a linear combination $L(z) = \sum_{j=1}^N c_j S(z,b_j)$ is as close to $H(z)$ in $A^\infty(\Omega)$ as desired. Now the function $g(z)$ given by

$$f_a(z)^{-1} \sum_{j=1}^N c_j S(z,b_j)/L(z,a)$$

extends to be a meromorphic function on the double of Ω which is C^∞ close to the identity function near and up to the boundary of Ω. We shall now make some adjustments to this function to eliminate any poles that might occur at the zeros of f_a. Since the complex span $\{S(z,b) : b \in D_\epsilon(w_0)\}$ is dense in $A^\infty(\Omega)$, there exist points B_k in $D_\epsilon(w_0)$ such that $\det[M_{jk}] \neq 0$ where $[M_{jk}]$ is the $n \times n$ matrix given by $M_{jk} = S(a_j, B_k)$ in which the indices range over $j = 0, 1, \ldots n-1$ and $k = 0, 1, \ldots, n-1$. Since $H(z)$ vanishes at the zeroes a_j, $j = 0, 1, \ldots n-1$, of f_a, the complex numbers $L(a_j)$ are small, and the closer $L(z)$ is to $H(z)$ in $A^\infty(\Omega)$, the smaller they are. Let μ_{jk} solve the system

$$L(a_j) = \sum_{k=0}^{n-1} \mu_{jk} S(a_j, B_k),$$

for $j = 0, 1, \ldots n-1$. Note that the complex numbers μ_{jk} are small and that they go to zero as L tends to H in $A^\infty(\Omega)$. We now revise the definition of the function $g(z)$ to be

$$f_a(z)^{-1} \left(\sum_{j=1}^N c_j S(z,b_j)/L(z,a) - \sum_{k=0}^{n-1} \mu_{jk} S(z,B_k)/L(z,a) \right).$$

This function has the virtue that it has no poles at the zeroes of f_a, and because it is C^∞ close to the identity near the boundary of Ω, it is close to the identity in $C^\infty(\overline{\Omega})$. Furthermore, the extension of this function to the reflected side in the double is given by the conjugate of

$$f_a(z) \left(\sum_{j=1}^N c_j L(z,b_j)/S(z,a) - \sum_{k=0}^{n-1} \mu_{jk} L(z,B_k)/S(z,a) \right),$$

(where we are thinking $z = R(\zeta)$ where R is the reflection function on the double). This function has only simple poles at the points b_j and B_k in $D_\epsilon(w_0)$. This completes the first part of the proof of Theorem 1.7. To prove the last assertion in the statement of Theorem 1.7, repeat the argument above, noting that the same reasoning shows that the complex linear span of $\{S^{(m)}(z,w_0), m = 0, 2, \ldots\}$ where $S^{(m)}(z,w) = (\partial^m/\partial\bar{w}^m)S(z,w)$ is also dense in $A^\infty(\Omega)$, and also observing that identity (2.2) can be used in the same way to show that $S^{(m)}(z,w_0)/L(z,a)$ extends meromorphically to the double.

4. How to zip the Bergman kernel. Suppose that Ω is a bounded n-connected domain whose boundary consists of n non-intersecting C^∞ smooth simple closed curves. Let $g(z)$ denote a Gustafsson function as constructed in the proofs of Theorems 1.6 or 1.7. Let $\widehat{\Omega}$ denote the double of Ω and let $R(z)$ denote the antiholomorphic reflection function which maps Ω to its reflected copy $\widetilde{\Omega}$. Let $G(z)$ denote the meromorphic extension of $g(z)$ to the double. Gustafsson [12] proved that $G(z)$ and $\overline{G(R(z))}$ form a primitive pair for the field of meromorphic functions on $\widehat{\Omega}$. Now $g(\Omega)$ is a quadrature domain and the function $g(z)$ transforms to be the function z on $g(\Omega)$. Hence, z and $\overline{G(R(g^{-1}(z)))}$ extend meromorphically to the double of $g(\Omega)$ and form a primitive pair. Let $h(z)$ denote the meromorphic function $\overline{G(R(g^{-1}(z)))}$. Notice that $h(z)$ is equal to $\bar{z}$ on $b\Omega$ and that h extends C^∞ smoothly up to the boundary.

Let $\{w_j\}_{j=1}^N$ denote the finitely many poles of $h(z)$ in Ω and let n_j be equal to the order of the pole at w_j. The numbers N, w_j, and n_j are exactly the numbers that appear in (1.1) in the quadrature identity for $g(\Omega)$. Let $P_j(z)$ denote the principal part of $h(z)$ at w_j. Theorem 1.1 yields that the Bergman kernel associated to $g(\Omega)$ is a rational combination of z and $h(z)$. We don't need to zip the function z. Recall that $h(z) = \bar{z}$ on the boundary of $g(\Omega)$, and so the function $h(z)$ can be zipped via the formula

$$h(z) - \sum_{j=1}^N P_j(z) = \frac{1}{2\pi i}\int_{b\Omega} \frac{\bar{\zeta} - \sum_{j=1}^N P_j(\zeta)}{\zeta - z}\, d\zeta.$$

But $\int_{b\Omega} \frac{1}{(\zeta-w_j)^k(\zeta-z)}\, d\zeta$ is zero for positive integers k. Hence

$$h(z) = \sum_{j=1}^N P_j(z) + \frac{1}{2\pi i}\int_{b\Omega} \frac{\bar{\zeta}}{\zeta - z}\, d\zeta.$$

Define $\mathcal{Q}$ via

$$\mathcal{Q}(z) = \frac{1}{2\pi i}\int_{b\Omega} \frac{\bar{\zeta}}{\zeta - z}\, d\zeta. \tag{4.1}$$

We may state that $\mathcal{Q}$ is a holomorphic function on Ω which extends meromorphically to the double of Ω without poles in $\overline{\Omega}$ and that $\mathcal{Q}$ is an algebraic function. We have just proved that the Bergman kernel associated to $g(\Omega)$ is a rational combination of z, $\mathcal{Q}(z)$, $\bar{w}$, and $\overline{\mathcal{Q}(w)}$. A rational function is encoded by finitely many complex coefficients and a few positive integers. These numbers carry all the information that is needed to unzip the Bergman kernel for the quadrature domain $g(\Omega)$ via formula (4.1). (Gustafsson [12] proved that any quadrature domain of finite area can be expressed as $g(\Omega)$ for some such g and smooth Ω, and so this result can be easily generalized.)

We now turn to zipping the Bergman kernel $K(z,w)$ for Ω. Let $H(z)$ denote the function $\overline{G(R(z))}$, which is meromorphic on Ω and extends C^∞ smoothly up to $b\Omega$ and has boundary values equal to $\overline{g(z)}$. The transformation formula for the Bergman kernel under biholomorphic maps together with the form of the Bergman kernel for $g(\Omega)$ reveals that $K(z,w)$ is equal to $g'(z)\overline{g'(w)}$ times a rational function of $g(z)$, $H(z)$, $\overline{g(w)}$, and $\overline{H(w)}$. The Cauchy integral formula

$$g'(w) = \frac{1}{2\pi i}\int_{b\Omega} \frac{g(z)}{(z-w)^2}\, dz$$

allows us to obtain g' inside Ω from the boundary values of g, and of course g can be unzipped in the same manner. The function $H(z) = \overline{G(R(z))}$ can be recovered in a way to similar to how we handled h above. Let $P(z)$ denote the sum of the principal parts of $H(z)$. Then, as above,

$$H(z) = P(z) + \frac{1}{2\pi i} \int_{b\Omega} \frac{\overline{g(\zeta)}}{\zeta - z}\, d\zeta.$$

Hence we see that the Bergman kernel can be recovered from the boundary values of the single function g, assuming that finitely many coefficients from two rational functions are known.

5. Generalized quadrature domains. We shall call an n-connected domain Ω in the plane such that no boundary component is a point a *generalized quadrature domain* if there exist finitely many points $\{w_j\}_{j=1}^N$ in the domain, non-negative integers n_j, and finitely many continuous closed curves or curve segments σ_m in Ω such that complex numbers c_{jk} and b_m exist satisfying

$$\int_\Omega f\, dA = \sum_{j=1}^{N} \sum_{k=0}^{n_j} c_{jk} f^{(k)}(w_j) + \sum_{m=1}^{M} b_m \int_{\sigma_m} f(z)\, dz \tag{5.1}$$

for every function f in the Bergman space of square integrable holomorphic functions on Ω. Here, dA denotes Lebesgue area measure. As before, we shall also need to assume that the domain under study has *finite area.* The property of being a generalized quadrature domain and the conditions mentioned in Theorems 1.1-1.4 are tied together nicely in the following theorem.

Theorem 5.1. *Suppose that Ω is an n-connected domain in the plane of finite area such that no boundary component is a point. The following conditions are equivalent.*

1. *Ω is a generalized quadrature domain.*
2. *The Bergman kernel extends to the double of Ω as a meromorphic function, i.e., the Bergman kernel is generated by the restriction of two functions of one variable that form a primitive pair for the field of meromorphic functions on the double of Ω.*
3. *There exists a proper holomorphic mapping f of Ω onto the unit disc such that f' extends to the double of Ω as a meromorphic function.*
4. *The derivative of every proper holomorphic mapping of Ω onto the unit disc extends to the double of Ω as a meromorphic function.*
5. *Every function H on Ω that extends meromorphically to the double of Ω is such that H' also extends to the double of Ω.*

We have proved most of the equivalences in Theorem 5.1 in the proofs of Theorems 1.1-1.4. To finish the proof, we need only show that if f is a proper holomorphic mapping of Ω onto the unit disc such that f' extends meromorphically to the double of Ω, then Ω is a generalized quadrature domain. The condition that f' extends to the double means that there is a conformal map Φ from Ω to a bounded domain $\widetilde{\Omega}$ whose boundary consists of n simple closed real analytic curves, and

$f' \circ \Phi^{-1}$ extends to the double of $\widetilde{\Omega}$. Let φ denote the inverse of Φ. Since $f \circ \varphi$ is a proper holomorphic mapping of $\widetilde{\Omega}$ onto the unit disc, and since these two domains have real analytic boundary, the mapping $f \circ \varphi$ extends holomorphically past the boundary of $\widetilde{\Omega}$. Hence, the derivative $\varphi' \cdot (f' \circ \varphi)$ extends holomorphically also. Furthermore, the derivative of the extension does not vanish on the boundary. Since $f' \circ \varphi$ extends to the double of $\widetilde{\Omega}$, and since $\widetilde{\Omega}$ has real analytic boundary, it follows that $f' \circ \varphi$ extends holomorphically past the boundary of $\widetilde{\Omega}$. We conclude that φ' extends past the boundary of $\widetilde{\Omega}$ at all but the finitely many boundary points where $f‘ \circ \varphi$ might vanish, and at the vanishing points, φ maps the boundary of $\widetilde{\Omega}$ to a cusp-like boundary point of Ω. Thus we conclude that Ω must have piecewise real analytic boundary.

Next, to see that Ω is a generalized quadrature domain, let $z(t)$ parameterize one of the boundary curve segments of Ω. Since $\ln |f(z(t))| \equiv 1$, it follows by differentiating with respect to t that

$$\frac{f'(z(t))}{f(z(t))} z'(t) = - \left(\overline{f'(z(t))} / \overline{f(z(t))} \right) \overline{z'(t)}. \tag{5.2}$$

Let f_b be an Ahlfors map such that f and f_b generate the meromorphic functions on the double of Ω. Since f' extends to the double as a meromorphic functions, we know that $f' = R(f, f_b)$ for some rational function R. Now $f = 1/\overline{f}$ on the boundary of Ω since f maps the boundary into the unit circle. The same is true for f_b. Hence, (5.2) yields that

$$z'(t) = -1/R \left(1/\overline{f(z(t))}, 1/\overline{f_b(z(t))} \right) \left(\overline{f'(z(t))} / \overline{f(z(t))^2} \right) \overline{z'(t)},$$

i.e., that $dz = \overline{H(z)} d\bar{z}$ on $b\Omega$ where H extends meromorphically to Ω. Following Aharonov and Shapiro [1], Gustafsson shows in [12, p. 223] that this condition is equivalent to being a generalized quadrature domain. This completes the proof.

6. Quadrature domains with respect to arc length measure. An analogous theorem to Theorem 1.5 can be proved for smooth quadrature domains with respect to boundary arc length measure. Suppose Ω is a bounded n-connected domain in the plane bounded by n non-intersecting C^∞ simple closed curves. We say that Ω is a *quadrature domain with respect to arc length measure* if there exist finitely many points $\{w_j\}_{j=1}^N$ in the domain and non-negative integers n_j such that complex numbers c_{jk} exist satisfying

$$\int_{b\Omega} f \; ds = \sum_{j=1}^{N} \sum_{k=0}^{n_j} c_{jk} f^{(k)}(w_j) \tag{6.1}$$

for every function f in the Hardy space $H^2(b\Omega)$ of holomorphic functions on Ω with square integrable boundary values on $b\Omega$ with respect to arc length measure ds (see [2] for basic facts about $H^2(b\Omega)$). The techniques used in the previous sections can be adapted to replace the Runge theorems used by Gustafsson in the proofs of his more general results by density theorems in A^∞ for the Szegő kernel. Indeed, we can follow Gustafsson's argument in [13, p. 76] to the letter, noting that Gustafsson's function h can be taken to be a complex linear combination of functions of the

form $S(z,b)$ where b ranges over an open subset of Ω. This is because identity (2.2) shows that $S(z,b)^2 dz = -\overline{L(z,b)^2 d\bar{z}}$, and hence, $h\sqrt{dz}$ is a "half-order differential" where $h(z) = S(z,b)$. Similar reasoning reveals the same thing about complex linear combinations of such functions. We may now follow Gustafsson's argument, using the fact that the complex linear span of $\{S(z,b) : b \in \Omega\}$ is dense in $A^\infty(\Omega)$ in place of the Runge-type approximation theorem he uses. Gustafsson's functions f_j on page 77 can also be approximated in $A^\infty(\Omega)$ by functions in this linear span. In this way, we may construct a function h in $A^\infty(\Omega)$ such that h^2 has a single valued antiderivative g which is as close to the identity map in $C^\infty(\overline{\Omega})$ as we desire. This yields a quadrature domain with respect to arc length measure $g(\Omega)$ which is conformally equivalent to Ω and as C^∞ close to Ω as we desire.

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